Common zeros of the solutions of two differential equations

Asim Asiri *

June 18, 2011

Abstract

We consider two homogeneous linear differential equations and we use Nevanlinna theory to determine when the solutions of these differential equations can have the same zeros or nearly the same zeros.

Keywords: Nevanlinna theory, differential equations.

1 Introduction

In this paper, we will see some ways in which Nevanlinna theory can be used to study the solutions of complex differential equations. First of all, we are going to state some useful theorems needed to prove our new results.

In particular, we will take the equation

$$w'' + Pw = 0, \tag{1}$$

where P is a polynomial of degree n, as an object of study. It is well known that every solution of (1) is an entire function.

In 1955, Wittich [14] proved the following theorem.

Theorem 1.1 *If $f \not\equiv 0$ is a non-trivial solution of $w'' + Aw = 0$, with $A \not\equiv 0$ entire, then we have:*

*__Acknowledgement:__ The author would like to thank his supervisor Prof. Jim Langley for his support and guidance. Also, he would like to thank King Abdulaziz University for financial support of his PhD study.

(i) $T(r, A) = S(r, f)$.

(ii) *If f has finite order, then A is a polynomial.*

(iii) *If a is a non-zero complex number, then f takes the value a infinitely often, and in fact*

$$N\left(r, \frac{1}{f-a}\right) \backsim T(r, f).$$

Here we use standard notation of Nevanlinna theory from [7].

The following facts follow from the asymptotic representation for solutions of (1); we refer the reader to [2, 9] for details.

Theorem 1.2 *Let P be a polynomial of degree n, and let w be a non-trivial solution of the equation (1). Then, w has order of growth equal to $\frac{n+2}{2}$. Moreover, if w is a solution of (1) which has infinitely many zeros, then we have*

$$\liminf_{r\to\infty} \frac{N(r, \frac{1}{w})}{r^{(n+2)/2}} > 0. \tag{2}$$

We refer the reader to the book of Laine [10], the influential paper [2], and to [1, 3, 4, 5, 6, 11, 12, 13] for extensive results on the zeros of solutions of linear differential equations with entire coefficients.

In the next section, we consider two differential equations with solutions having the same or nearly the same zeros.

2 A theorem for the general case

To state our main result we need the following lemma.

Lemma 2.1 *Suppose $w'' = -Pw$ where P is a polynomial. Then for $j \geq 0$, there exist polynomials Q_j and R_j such that*

$$w^{(j)} = Q_j w + R_j w'. \tag{3}$$

Proof: In fact, we have the following initial cases:

$$\begin{aligned}
j = 0 \quad &\Rightarrow \quad Q_0 = 1, \quad R_0 = 0; \\
j = 1 \quad &\Rightarrow \quad Q_1 = 0, \quad R_1 = 1; \\
j = 2 \quad &\Rightarrow \quad Q_2 = -P, \quad R_2 = 0; \\
j = 3 \quad &\Rightarrow \quad Q_3 = -P', \quad R_3 = -P; \\
j = 4 \quad &\Rightarrow \quad Q_4 = P^2 - P'', \quad R_4 = -2P'.
\end{aligned} \tag{4}$$

Now we proceed by induction; assume that $j \geq 0$ and that (3) is true. Then we have

$$\begin{aligned} w^{(j+1)} &= Q'_j w + Q_j w' + R'_j w' + R_j w'' \\ &= Q'_j w + Q_j w' + R'_j w' - R_j P w \\ &= (Q'_j - R_j P) w + (Q_j + R'_j) w' \\ &= (Q_{j+1}) w + (R_{j+1}) w', \end{aligned}$$

where $Q_{j+1} = Q'_j - R_j P$ and $R_{j+1} = Q_j + R'_j$.
Since Q_{j+1} and R_{j+1} are polynomials, the induction is complete. □

Theorem 2.1 *Let $P \not\equiv 0$ be a polynomial of degree n. Let $w \not\equiv 0$ be a solution of (1). Assume that w has infinitely many zeros. Suppose that we have an entire solution $v \not\equiv 0$ of the differential equation*

$$v^{(k)} + \sum_{1 \leq j \leq k-2} B_j v^{(j)} + Av = 0, \quad k \geq 2, \tag{5}$$

such that A and B_j are entire functions with

$$\sum_{1 \leq j \leq k-2} T(r, B_j) + T(r, A) = o(r^{(n+2)/2}), \qquad \text{as } r \to \infty, \tag{6}$$

where the sum $\sum_{1 \leq j \leq k-2}$ should be interpreted as empty when $k = 2$. Assume that $N(r) = o(r^{(n+2)/2})$ where $N(r)$ counts both zeros and poles of $\frac{v}{w}$. Let

$$v = Lw. \tag{7}$$

Then one of the following possibilities holds.

(a) *L is constant, and*

$$A = -Q_k - \sum_{j=1}^{k-2} B_j Q_j, \tag{8}$$

where Q_k and Q_j are defined by Lemma 2.1.

(b) *L is not constant, but L satisfies*

$$\sum_{m=0}^{k} \left[\binom{k}{m} L^{(m)} R_{k-m} + \sum_{j=1}^{k-2} \binom{j}{m} B_j L^{(m)} R_{j-m} \right] = 0, \tag{9}$$

and A satisfies

$$A = -\left(\sum_{m=0}^{k} \left[\binom{k}{m} \frac{L^{(m)}}{L} Q_{k-m} + \sum_{j=1}^{k-2} \binom{j}{m} B_j \frac{L^{(m)}}{L} Q_{j-m} \right] \right), \tag{10}$$

where R_{k-m}, R_{j-m}, Q_{k-m} and Q_{j-m} are also defined by Lemma 2.1.

(c) *If $B_1, B_2, \cdots, B_{k-2}$ are polynomials, and case(b) holds, then A is a polynomial.*

There is no loss of generality in assuming that there is no term in w' in (1) and that there is no term B_{k-1} in (5). This is because for any equation

$$y^{(m)} + A_{m-1}y^{(m-1)} + \cdots + A_0 y = 0,$$

with entire coefficients A_j, the change of variables $y = UY$, where $mU'/U = -A_{m-1}$, gives an equation

$$Y^{(m)} + B_{m-2}Y^{(m-2)} + \cdots + B_0 Y = 0$$

and Y has the same zeros as y.

From Theorem 2.1 we will deduce the following results for the cases $k = 2$, $k = 3$ and $k = 4$.

Theorem 2.2 *Let $P \not\equiv 0$ be a polynomial of degree n. Let $w \not\equiv 0$ be a solution of (1). Assume that w has infinitely many zeros. Suppose that we have an entire solution $v \not\equiv 0$ of the differential equation*

$$v'' + Av = 0, \tag{11}$$

such that A is an entire function and $N(r)$ counts zeros of v which are not zeros of w and zeros of w which are not zeros of v. Assume that

$$N(r) + T(r, A) = o(r^{(n+2)/2}).$$

Then $\frac{v}{w}$ is a constant and $A = P$.

Example 2.1 *Obviously we may take $v = w$ and $A = P$.*

Example 2.2 *We give an example to show that $T(r, A) = o(r^{(n+2)/2})$ is necessary in Theorem 2.2. To show this put $v = we^g$ where g is an entire function. Then we have*

$$\begin{aligned}\frac{v''}{v} &= \frac{w''}{w} + 2g'\frac{w'}{w} + g'' + g'^2\\ &= -P + g'' + g'^2 + 2g'\frac{w'}{w}\\ &= -A.\end{aligned}$$

Now, if we put $g' = w$ then we get

$$-A = -P + w' + w^2 + 2w'.$$

So A is entire and

$$T(r,A) = O(r^{(n+2)/2}).$$

But, $\frac{v}{w} = e^g = e^{\int w}$ is not a constant. So, Theorem 2.2 is not true for $T(r,A) \neq o(r^{(n+2)/2})$.

Theorem 2.3 *Let $P \not\equiv 0$ be a polynomial of degree n. Let $w \not\equiv 0$ be a solution of (1). Assume that w has infinitely many zeros. Suppose that we have an entire solution $v \not\equiv 0$ of the differential equation*

$$v''' + Av = 0, \tag{12}$$

such that A is an entire function with $T(r,A) = o(r^{(n+2)/2})$ and $N(r) = o(r^{(n+2)/2})$, where $N(r)$ counts both zeros and poles of $\frac{v}{w}$. Then $v = Lw$, where $L'' = \frac{P}{3}L$ and $A = \frac{2}{3}P' + \frac{8}{3}P\frac{L'}{L}$ is a polynomial.

Example 2.3 *The exceptional case in the conclusion can occur in Theorem 2.3. For example, take $L = e^Q$ where Q is a polynomial, and set*

$$\frac{P}{3} = \frac{L''}{L} = Q'^2 + Q'',$$

so that P is a polynomial. Then

$$L'' = \frac{P}{3}L, \qquad L''' = \frac{P}{3}L' + \frac{P'}{3}L.$$

If w solves (1) then $v = Lw$ satisfies

$$v''' = L(-P'w - Pw') + 3L'(-Pw) + 3\frac{P}{3}Lw' + \left(\frac{P}{3}L' + \frac{P'}{3}L\right)w$$

and so v solves (12) with

$$\begin{aligned} A &= \frac{2}{3}P' + \frac{8}{3}P\frac{L'}{L} \\ &= \frac{2}{3}P' + \frac{8}{3}PQ', \end{aligned}$$

which is also a polynomial.

Theorem 2.4 *Let $P \not\equiv 0$ be a polynomial of degree n. Let $w \not\equiv 0$ be a solution of (1). Assume that w has infinitely many zeros. Suppose that we have an entire solution $v \not\equiv 0$ of the differential equation*

$$v''' + Bv' + Av = 0, \tag{13}$$

such that A and B are entire functions with $T(r,A) + T(r,B) = o(r^{(n+2)/2})$. Assume that $N(r) = o(r^{(n+2)/2})$, where $N(r)$ counts both zeros and poles of $\frac{v}{w}$. Then $v = Lw$ and one of the following holds.

(a) *L is constant and $A = P'$, $B = P$.*

(b) *L is non-constant and $L'' = \frac{1}{3}PL - \frac{1}{3}BL$ and*

$$A = \frac{8}{3}P\frac{L'}{L} + \frac{2}{3}P' + \frac{1}{3}B' - \frac{2}{3}B\frac{L'}{L}.$$

Example 2.4 *To show that case (b) can occur in Theorem 2.4, we can use Example 2.3, with $B = 0$.*

Example 2.5 *We give an example to show that $T(r, A) + T(r, B) = o(r^{(n+2)/2})$ is necessary in Theorem 2.4. To show this put $v = we^g$ where g is an entire function. Then we have*

$$\begin{aligned}\frac{v'''}{v} &= \frac{w'''}{w} + 3\frac{w''}{w}g' + 3\frac{w'}{w}\left(g'' + g'^2\right) + g'^3 + 3g'g'' + g''' \\ &= -P' - P\frac{w'}{w} - 3Pg' + 3\frac{w'}{w}\left(g'' + g'^2\right) + g'^3 + 3g'g'' + g''' \\ &= -B\frac{v'}{v} - A.\end{aligned}$$

Now, if we put $g' = w$ then we get

$$-B\frac{v'}{v} - A = -P' - P\frac{w'}{w} - 3Pw + 3\frac{w'}{w}\left(w' + w^2\right) + w^3 + 3ww' + w''.$$

So

$$-B\left(\frac{w'}{w} + g'\right) - A = -P' - P\frac{w'}{w} - 3Pw + 3\frac{w'^2}{w} + 3w'w + w^3 + 3ww' + w''.$$

Then

$$-B\left(\frac{w'}{w} + w\right) - A = -P' - P\frac{w'}{w} - 3Pw + 3\frac{w'^2}{w} + 3ww' + w^3 + 3ww' + w''.$$

We want A to be entire. Put $-B = -P + 3w'$, then

$$-A = -P' - 2Pw + 3ww' + w^3 + w''$$

and since $w'' = -Pw$, then we get

$$A = P' + 3Pw - 3ww' - w^3. \tag{14}$$

Then A and B are entire functions. So

$$T(r, A) + T(r, B) = O(r^{(n+2)/2}).$$

But, $\frac{v}{w} = e^g = e^{\int w}$ is not a constant. This shows that case (a) does not hold.

Now, we check whether case (b) holds or not. From case (b) we have

$$\begin{aligned}
A &= \frac{8}{3}P\frac{L'}{L} + \frac{2}{3}P' + \frac{1}{3}B' - \frac{2}{3}B\frac{L'}{L} \\
&= \frac{8}{3}Pw + \frac{2}{3}P' + \frac{1}{3}(P' - 3w'') - \frac{2}{3}(P - 3w')w \\
&= \frac{8}{3}Pw + \frac{2}{3}P' + \frac{P'}{3} + Pw - \frac{2}{3}Pw + 2ww' \\
&= 3Pw + P' + 2ww'.
\end{aligned}$$

But this and (14) *are not the same, because if they are, then*

$$P' + 3Pw - 3ww' - w^3 = 3Pw + P' + 2ww'$$

and so

$$w^3 = -5ww'.$$

Dividing by w^3 *gives*

$$1 = -5\frac{w'}{w^2},$$

and by integrating we can write

$$z + c = \frac{-5}{w}, \qquad \text{where } c \text{ is a constant,}$$

which is impossible since w *is transcendental entire. So case (b) also does not hold.*

Therefore, Theorem 2.4 is not true for $T(r, A) + T(r, B) \neq o(r^{(n+2)/2})$.

Theorem 2.5 *Let* $P \not\equiv 0$ *be a polynomial of degree* n. *Let* $w \not\equiv 0$ *be a solution of* (1). *Assume that* w *has infinitely many zeros. Suppose that we have an entire solution* $v \not\equiv 0$ *of the differential equation*

$$v^{(4)} + Av = 0, \tag{15}$$

such that A *is an entire function with* $T(r, A) = o(r^{(n+2)/2})$ *and* $N(r) = o(r^{(n+2)/2})$, *where* $N(r)$ *counts both zeros and poles of* $\frac{v}{w}$. *Then* $v = Lw$ *and one of the following holds.*

(a) *L is constant and so are P and A.*

(b) *L is non-constant,* $S = \frac{L'}{L}$ *is a rational function, and* $P = S^2 + 2S'$, *while*

$$A = 5P\frac{L''}{L} + \frac{5}{2}P'\frac{L'}{L} + \frac{1}{2}P'' - P^2$$

and A *is a polynomial.*

Example 2.6 *To show that case (b) can occur in Theorem 2.5, take* $L = Y^2 = e^Q$ *where* Q *is a polynomial and set*

$$Q' = S = 2y = 2\frac{Y'}{Y}, \qquad P = S^2 + 2S' = 4(y^2 + y')$$

so that P *is a polynomial. Then*

$$\begin{aligned} L' &= 2YY', \\ L'' &= 2Y'^2 + 2YY'' = 2Y'^2 + \frac{P}{2}L, \\ L''' &= PL' + \frac{P'}{2}L, \\ L^{(4)} &= PL'' + \frac{3}{2}P'L' + \frac{P''}{2}L. \end{aligned}$$

If w *solves* (1) *then* $v = Lw$ *satisfies*

$$\begin{aligned} v^{(4)} = L\left((P^2 - P'')w - 2P'w'\right) + 4L'(-Pw' - P'w) + 6L''(-Pw) \\ + 4\left(PL' + \frac{P'}{2}L\right)w' + \left(PL'' + \frac{3}{2}P'L' + \frac{P''}{2}L\right)w \end{aligned}$$

and so v *solves* (15) *with*

$$\begin{aligned} A &= 5P\frac{L''}{L} + \frac{5}{2}P'\frac{L'}{L} + \frac{1}{2}P'' - P^2 \\ &= 5P(Q'^2 + Q'') + \frac{5}{2}P'Q' + \frac{1}{2}P'' - P^2, \end{aligned}$$

which is also a polynomial.

3 Proof of Theorem 2.1

Let w and v be as in the hypotheses. Since w has infinitely many zeros, then by Theorem 1.2 we have (2).

Claim A: We claim that w has simple zeros and

$$N\left(r, \frac{1}{w}\right) = N\left(r, \frac{w'}{w}\right).$$

This holds by the existence-uniqueness theorem [9].

From equation (1) and Lemma 2.1, we have (3).

From (3), (7) and by using Leibniz' rule, we get, for $1 \leq j \leq k$,

$$
\begin{aligned}
v^{(j)} &= \sum_{m=0}^{j} \binom{j}{m} L^{(m)} w^{(j-m)} \\
&= \sum_{m=0}^{j} \binom{j}{m} L^{(m)} \left(Q_{j-m} w + R_{j-m} w'\right) \qquad \text{(by Lemma 2.1)} \\
&= \left(\sum_{m=0}^{j} \binom{j}{m} L^{(m)} Q_{j-m} \right) w + \left(\sum_{m=0}^{j} \binom{j}{m} L^{(m)} R_{j-m} \right) w'. \qquad (16)
\end{aligned}
$$

From (5) and (16), we find that

$$
\begin{aligned}
-Av &= -ALw \\
&= v^{(k)} + \sum_{1 \leq j \leq k-2} B_j v^{(j)} \\
&= \left(\sum_{m=0}^{k} \binom{k}{m} L^{(m)} Q_{k-m} \right) w + \left(\sum_{m=0}^{k} \binom{k}{m} L^{(m)} R_{k-m} \right) w' \\
&\quad + \sum_{j=1}^{k-2} B_j \left[\left(\sum_{m=0}^{j} \binom{j}{m} L^{(m)} Q_{j-m} \right) w + \left(\sum_{m=0}^{j} \binom{j}{m} L^{(m)} R_{j-m} \right) w' \right].
\end{aligned}
$$

Now, we can write, for $1 \leq j \leq k-2$,

$$
\sum_{m=0}^{j} \binom{j}{m} L^{(m)} Q_{j-m} = \sum_{m=0}^{k} \binom{j}{m} L^{(m)} Q_{j-m},
$$

and

$$
\sum_{m=0}^{j} \binom{j}{m} L^{(m)} R_{j-m} = \sum_{m=0}^{k} \binom{j}{m} L^{(m)} R_{j-m},
$$

because $\binom{j}{m} = 0$ when $j < m \leq k$.

So,

$$
\begin{aligned}
-ALw &= \left[\sum_{m=0}^{k}\binom{k}{m}L^{(m)}Q_{k-m} + \sum_{j=1}^{k-2}B_j\left(\sum_{m=0}^{k}\binom{j}{m}L^{(m)}Q_{j-m}\right)\right]w \\
&\quad + \left[\sum_{m=0}^{k}\binom{k}{m}L^{(m)}R_{k-m} + \sum_{j=1}^{k-2}B_j\left(\sum_{m=0}^{k}\binom{j}{m}L^{(m)}R_{j-m}\right)\right]w' \\
&= \left[\sum_{m=0}^{k}\binom{k}{m}L^{(m)}Q_{k-m} + \sum_{m=0}^{k}\sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}Q_{j-m}\right]w \\
&\quad + \left[\sum_{m=0}^{k}\binom{k}{m}L^{(m)}R_{k-m} + \sum_{m=0}^{k}\sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}R_{j-m}\right]w' \\
&= \left(\sum_{m=0}^{k}\left[\binom{k}{m}L^{(m)}Q_{k-m} + \sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}Q_{j-m}\right]\right)w \\
&\quad + \left(\sum_{m=0}^{k}\left[\binom{k}{m}L^{(m)}R_{k-m} + \sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}R_{j-m}\right]\right)w'.
\end{aligned}
$$

Then we get

$$
\begin{aligned}
0 &= \left(\sum_{m=0}^{k}\left[\binom{k}{m}L^{(m)}Q_{k-m} + \sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}Q_{j-m}\right] + AL\right)w \\
&\quad + \left(\sum_{m=0}^{k}\left[\binom{k}{m}L^{(m)}R_{k-m} + \sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}R_{j-m}\right]\right)w',
\end{aligned}
$$

and so

$$
\begin{aligned}
0 &= \sum_{m=0}^{k}\left[\binom{k}{m}L^{(m)}Q_{k-m} + \sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}Q_{j-m}\right] + AL \\
&\quad + \left(\sum_{m=0}^{k}\left[\binom{k}{m}L^{(m)}R_{k-m} + \sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}R_{j-m}\right]\right)\frac{w'}{w}. \qquad (17)
\end{aligned}
$$

Now, we have three cases.

Case (I): If L is a constant, then w solves (5) and, by using Lemma 2.1, we get the following equations

$$
\begin{cases} w^{(k)} + \sum_{1\le j\le k-2}B_jw^{(j)} + Aw = 0, \\ -w^{(k)} + Q_kw + R_kw' = 0. \end{cases}
$$

By adding these two equations and using (3) again, we obtain

$$\begin{aligned} 0 &= \sum_{1\le j\le k-2} B_j\left(Q_j w + R_j w'\right) + Aw + Q_k w + R_k w' \\ &= \left(A + Q_k + \sum_{1\le j\le k-2} B_j Q_j\right) w + \left(R_k + \sum_{1\le j\le k-2} B_j R_j\right) w'. \end{aligned}$$

Then, we get

$$\left(R_k + \sum_{1\le j\le k-2} B_j R_j\right)\frac{w'}{w} + A + Q_k + \sum_{1\le j\le k-2} B_j Q_j = 0.$$

Now, if $R_k + \sum_{1\le j\le k-2} B_j R_j \equiv 0$, then $A + Q_k + \sum_{1\le j\le k-2} B_j Q_j \equiv 0$ and so we have (8) and conclusion (a).

Suppose next that $R_k + \sum_{1\le j\le k-2} B_j R_j \not\equiv 0$; then

$$w = 0 \Rightarrow \frac{w'}{w} = \infty \Rightarrow R_k + \sum_{1\le j\le k-2} B_j R_j = 0.$$

Recall that all zeros of w are simple. We deduce that

$$\begin{aligned} N\left(r, \frac{1}{w}\right) &\le N\left(r, \frac{1}{R_k + \sum_{1\le j\le k-2} B_j R_j}\right) \\ &\le T\left(r, R_k + \sum_{1\le j\le k-2} B_j R_j\right) + O(1) \\ &= o\left(r^{(n+2)/2}\right). \end{aligned}$$

But this contradicts (2).

Case (II): Suppose that L is not constant and (9) holds. Then from (17) we get (10) and conclusion (b) of the theorem.

Suppose in addition that $B_1, B_2, \cdots, B_{k-2}$ are polynomials. Since $R_0 = 0$ and $R_1 = 1$ in (4) we see from (9) that L satisfies a homogenous linear differential equation of order $k-1$ with polynomial coefficients, and so L has finite order. Furthermore, (10) and the lemma of the logarithmic derivative now give $T(r, A) = m(r, A) = O(\log r)$, so that A is a polynomial. This completes the discussion of Case (II) and the proof of part (c) of the theorem.

It remains only to show that the following case is impossible.

Case (III): Supposed that L is not constant and (9) does not hold, that is

$$\sum_{m=0}^{k}\left[\binom{k}{m}L^{(m)}R_{k-m}+\sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}R_{j-m}\right]\not\equiv 0.$$

Let $S=L'/L$. We first compare $N(r,S)$ with $N(r)$. Recall that all zeros of w are simple. On the other hand, v solves a differential equation of order k. So, zeros of v have multiplicities less than or equal to $k-1$.

So, $L=\frac{v}{w}$ has zeros with multiplicities at most $k-1$, and has simple poles. Then, we have

$$\begin{aligned} N(r,S) &= \overline{N}\left(r,\frac{1}{L}\right)+\overline{N}(r,L) \\ &\le N\left(r,\frac{1}{L}\right)+N(r,L) \\ &= N(r) \qquad (18)\\ &\le (k-1)\overline{N}\left(r,\frac{1}{L}\right)+\overline{N}(r,L) \\ &\le (k-1)N(r,S). \end{aligned}$$

Claim 1: We claim that

$$T(r,S)\le o\left(r^{(n+2)/2}\right)$$

for r outside a set E of finite linear measure.

To prove this, we use the fact that $Q_0=1$ and $R_0=0$ in Lemma 2.1 to write (17) in the form

$$\begin{aligned} 0 &= \frac{L^{(k)}}{L}+A \\ &+\sum_{m=0}^{k-1}\frac{L^{(m)}}{L}\left[\binom{k}{m}\left(Q_{k-m}+R_{k-m}\frac{w'}{w}\right)+\sum_{j=1}^{k-2}\binom{j}{m}B_j\left(Q_{j-m}+R_{j-m}\frac{w'}{w}\right)\right]. \end{aligned} \qquad (19)$$

We can write, for $1\le m\le k$,

$$\frac{L^{(m)}}{L}=S^m+U_{m-1}(S),$$

where $U_{m-1}(S)$ is a polynomial in $S,S',S'',\cdots,S^{(k)}$ with constant coefficients and total degree at most $m-1$. This follows immediately from Lemma 3.5 in [7], and is easily proved by induction.

This gives us an integer $q > 0$ such that (19) may be written as

$$S^k = \sum_{j=0}^{q} \left(a_j + b_j \frac{w'}{w} \right) S^{i_{0,j}} \left(S'\right)^{i_{1,j}} \left(S''\right)^{i_{2,j}} \cdots \left(S^{(k)}\right)^{i_{k,j}}, \tag{20}$$

where $i_{\mu,j} \geq 0$ are integers and

$$\sum_{\mu=0}^{k} i_{\mu,j} \leq k-1$$

for each j. Here a_j and b_j are polynomials in A, B_μ, Q_μ and R_μ, and so satisfy

$$m(r, a_j) + m(r, b_j) = o\left(r^{(n+2)/2}\right) \qquad \text{as } r \to \infty.$$

By Clunie's lemma [10, p. 39] we obtain

$$m(r, S) \leq o\left(r^{(n+2)/2}\right) + O(\log^+ T(r, S)) \tag{21}$$

for r outside a set E of finite linear measure.

Now, we use (18) and (21) to get

$$\begin{aligned} T(r, S) &= N(r, S) + m(r, S) \\ &\leq N(r) + m(r, S) \\ &\leq o\left(r^{(n+2)/2}\right) + O\left(\log^+ T(r, S)\right) \end{aligned}$$

and so

$$T(r, S) = o\left(r^{(n+2)/2}\right)$$

for r outside a set E of finite linear measure. This proves Claim 1.

Claim 2: We claim that

$$T(r, S) \leq o\left(r^{(n+2)/2}\right) \qquad \text{for all large } r.$$

This follows from [10, Lemma 1.1.1].

Now, dividing (17) by L shows that if $\frac{w'}{w}$ has a pole at z then either

$$A_2 = \sum_{m=0}^{k} \left[\binom{k}{m} \frac{L^{(m)}}{L} R_{k-m} + \sum_{j=1}^{k-2} \binom{j}{m} B_j \frac{L^{(m)}}{L} R_{j-m} \right] = 0$$

at z or

$$A_1 = \sum_{m=0}^{k} \left[\binom{k}{m} \frac{L^{(m)}}{L} Q_{k-m} + \sum_{j=1}^{k-2} \binom{j}{m} B_j \frac{L^{(m)}}{L} Q_{j-m} \right] + A = \infty$$

at z.

Once we have Claim 2, we can write (17) as

$$A_1 + A_2\frac{w'}{w} = 0, \tag{22}$$

where $T(r, A_j) = o\left(r^{(n+2)/2}\right)$, $j = 1, 2$ and $A_2 \not\equiv 0$ by the assumption of Case (III).

Now, by using Claim A and (22), we get

$$\begin{aligned} N\left(r, \frac{1}{w}\right) &= N\left(r, \frac{w'}{w}\right) \\ &\leq N\left(r, \frac{1}{A_2}\right) + N(r, A_1) \\ &\leq T(r, A_2) + T(r, A_1) + O(1) \\ &\equiv o\left(r^{(n+2)/2}\right). \end{aligned}$$

Hence,

$$\lim_{r\to\infty} \frac{N\left(r, \frac{1}{w}\right)}{r^{(n+2)/2}} = 0.$$

But this contradicts (2). Therefore, Case (III) cannot occur. □

4 Proof of Theorem 2.2

Assume the hypotheses of Theorem 2.2. Taking $k = 2$ in Theorem 2.1, we have two cases to consider.

Case (a): L is a constant and

$$A = -Q_2 = -(-P) = P \qquad \text{(by using (8) and Lemma 2.1).}$$

Case (b): L is not constant, but

$$\begin{aligned} 0 &= \sum_{m=0}^{2} \binom{2}{m} L^{(m)} R_{2-m} \\ &= LR_2 + 2L'R_1 + L''R_0 \\ &= 2L' \qquad \text{by using Lemma 2.1.} \end{aligned}$$

But this implies that L is constant, a contradiction. □

5 Proof of Theorem 2.4

Assume the hypotheses of Theorem 2.4. Taking $k = 3$ and $B_1 = B$ in Theorem 2.1, we have two cases to consider.

Case (a): L is a constant and

$$A = -Q_3 - B_1Q_1 = -Q_3 = P' \qquad \text{(by using (8) and Lemma 2.1).}$$

But, since w solves (13) we have

$$w''' + Bw' + Aw = 0$$

and so

$$w''' + Bw' + P'w = 0. \tag{23}$$

Also, by differentiating (1) we get

$$w''' + P'w + Pw' = 0. \tag{24}$$

Now, (23) and (24) give $P = B$.

Case (b): L is not constant and

$$\begin{aligned} 0 &= \sum_{m=0}^{3}\left[\binom{3}{m}L^{(m)}R_{3-m} + \binom{1}{m}B_1L^{(m)}R_{1-m}\right] \\ &= LR_3 + B_1L + 3L'R_2 + 0 + 3L''R_1 + 0 + L'''R_0 + 0 \\ &= -PL + BL + 3L'' \end{aligned}$$

and so

$$L'' = \frac{1}{3}PL - \frac{1}{3}BL. \tag{25}$$

Differentiating (25) we get

$$L''' = \frac{1}{3}P'L + \frac{1}{3}PL' - \frac{1}{3}B'L - \frac{1}{3}BL'. \tag{26}$$

Also, we have

$$\begin{aligned} -A &= \sum_{m=0}^{3}\left[\binom{3}{m}\frac{L^{(m)}}{L}Q_{3-m} + \binom{1}{m}B_1\frac{L^{(m)}}{L}Q_{1-m}\right] \\ &= [Q_3 + 0] + \left[3\frac{L'}{L}Q_2 + B_1\frac{L'}{L}\right] + \left[3\frac{L''}{L}Q_1 + 0\right] + \left[\frac{L'''}{L}Q_0\right] \\ &= -P' - 3P\frac{L'}{L} + B\frac{L'}{L} + \frac{L'''}{L}. \end{aligned}$$

Thus,
$$A = P' + 3P\frac{L'}{L} - B\frac{L'}{L} - \frac{L'''}{L}.$$
Using (26) we get
$$A = P' + 3P\frac{L'}{L} - B\frac{L'}{L} - \frac{1}{3}P' - \frac{1}{3}P\frac{L'}{L} + \frac{1}{3}B' + \frac{1}{3}B\frac{L'}{L}$$
and so
$$A = \frac{8}{3}P\frac{L'}{L} + \frac{2}{3}P' + \frac{1}{3}B' - \frac{2}{3}B\frac{L'}{L}.$$
□

6 Proof of Theorem 2.3

We will deduce Theorem 2.3 from Theorem 2.4, since (12) is just (13) with $B = 0$.

Assume the hypotheses of Theorem 2.3. Then v and w satisfy conclusion (a) or (b) of Theorem 2.4 with $B = 0$. But conclusion (a) gives $P = B = 0$, which is impossible, and so we must have conclusion (b). Since $B = 0$ this gives $L'' = \frac{P}{3}L$ and $A = \frac{2}{3}P' + \frac{8}{3}P\frac{L'}{L}$ as asserted, and A is a polynomial by part (c) of Theorem 2.1. □

7 Proof of Theorem 2.5

Assume the hypotheses of Theorem 2.5. Taking $k = 4$ and $B_1 = B_2 = 0$ in Theorem 2.1, we have two cases to consider.

Case (a): L is a constant and
$$A = -Q_4 = -P^2 + P'' \qquad \text{by using (4).}$$
But, differentiating (1) two times gives
$$\begin{aligned} 0 &= w^{(4)} + P''w + 2P'w' + Pw'' \\ &= w^{(4)} + (P'' - P^2)w + 2P'w' \end{aligned}$$
Since we also have $w^{(4)} + Aw = 0$, this gives
$$0 = 2P'w'$$
and so P must be constant and so must A.

Case (b): L is non-constant and L satisfies, using (4),

$$\begin{aligned} 0 &= \sum_{m=0}^{4} \binom{4}{m} L^{(m)} R_{4-m} \\ &= LR_4 + 4L'R_3 + 6L''R_2 + 4L'''R_1 + L^{(4)}R_0 \\ &= -2LP' - 4L'P + 0 + 4L''' + 0 \\ &= 4L''' - 4L'P - 2LP' \end{aligned}$$

and so

$$L''' = L'P + \frac{1}{2}LP'. \tag{27}$$

Since this is a linear differential equation and P is a polynomial it follows that L is an entire function.

We write (27) in the form

$$P' + 2\frac{L'}{L}P = 2\frac{L'''}{L}. \tag{28}$$

It is then elementary to show that

$$P = 2\frac{L''}{L} - \left(\frac{L'}{L}\right)^2 + \frac{c}{L^2} \tag{29}$$

with c a constant. Also L is not a polynomial, since $P(\infty) \neq 0$.

Claim 1: We claim that $\rho(L) = (n+2)/2$ and

$$\liminf_{r\to\infty} \frac{T(r,L)}{r^{(n+2)/2}} > 0. \tag{30}$$

To prove this we use Wiman-Valiron theory [8]. Take z_0 such that $|z_0| = r$, $|L(z_0)| = M(r,L)$ and $r \notin E$, where E is the exceptional set of finite logarithmic measure. Then,

$$\frac{L'}{L}(z_0) \sim \frac{\nu(r,L)}{z_0},$$

where $\nu(r,L)$ is the central index, and

$$\frac{L''}{L}(z_0) \sim \frac{\nu(r,L)^2}{z_0^2} \qquad , \qquad \frac{c}{L(z_0)^2} = \frac{o(1)}{z_0^2}.$$

Now, we have, for some $c_1 \neq 0$,

$$\begin{aligned} c_1 z_0^n \sim P(z_0) &= 2\frac{\nu(r,L)^2}{z_0^2}(1+o(1)) - \frac{\nu(r,L)^2}{z_0^2}(1+o(1)) + \frac{o(1)}{z_0^2} \\ &= (1+o(1))\frac{\nu(r,L)^2}{z_0^2}. \end{aligned}$$

So,

$$\nu(r,L)^2 \sim c_1 z_0^{(n+2)}$$

and so using c_j to denote non-zero constants,

$$\nu(r,L) \sim c_2 r^{(n+2)/2} \qquad (r \notin E).$$

Therefore,

$$\nu(r,L) \sim c_2 r^{(n+2)/2} \qquad (\text{ for all } r \to \infty),$$

since we may choose $r' < r < r''$ with $r' \sim r \sim r''$ and $r', r'' \notin E$. So the maximum term $\mu(r,L)$ satisfies

$$\begin{aligned}\log \mu(r,L) &= c_3 + \int_{c_4}^{r} \nu(r,L)\frac{dt}{t} \\ &\sim c_5 r^{(n+2)/2}.\end{aligned}$$

Also,

$$\mu(r,L) \leq M(r,L) \leq 2\mu(2r,L).$$

This gives

$$\begin{aligned}c_5 r^{(n+2)/2} \sim \log \mu(r,L) &\leq \log M(r,L) \\ &\leq \log \mu(2r,L) + \log 2 \\ &\leq c_6 r^{(n+2)/2}.\end{aligned}$$

Similarly, we have

$$T(r,L) \leq \log M(r,L) \leq 3T(2r,L).$$

So

$$T(r,L) \leq c_6 r^{(n+2)/2}$$

and

$$T(r,L) \geq \frac{1}{3}\log M(\frac{r}{2},L) \geq c_7 r^{(n+2)/2}.$$

This leads to $\rho(L) = (n+2)/2$ and (30) which completes the proof of Claim 1.

Claim 2: We claim that $c = 0$ in (29). If this is not the case then (29) and the lemma of the logarithmic derivative give

$$m\left(r, \frac{1}{L}\right) = O(\log r).$$

But then

$$\begin{aligned} T(r, L) &= T\left(r, \frac{1}{L}\right) + O(1) \\ &\leq m\left(r, \frac{1}{L}\right) + N\left(r, \frac{1}{L}\right) + O(1) \\ &= o(r^{(n+2)/2}), \end{aligned}$$

which contradicts (30).

Hence, $c = 0$ in (29) as asserted and this completes the proof of Claim 2.

Now L is entire, and we write, locally, $L = Y^2$ and $S = \frac{L'}{L} = 2\frac{Y'}{Y} = 2y$. Then (29) gives

$$\begin{aligned} P &= 2(S^2 + S') - S^2 \\ &= S^2 + 2S' \\ &= 4(y^2 + y') \end{aligned}$$

and so

$$Y'' = \frac{P}{4} Y.$$

Hence, Y is an entire function. But

$$N\left(r, \frac{1}{Y}\right) = \frac{1}{2} N\left(r, \frac{1}{L}\right) = o\left(r^{(n+2)/2}\right)$$

and so Y has finitely many zeros, by Theorem 1.2. Hence y and S are rational functions and A satisfies, using (4) and (10),

$$\begin{aligned} -A &= \sum_{m=0}^{4} \binom{4}{m} \frac{L^{(m)}}{L} Q_{4-m} \\ &= Q_4 + 4\frac{L'}{L} Q_3 + 6\frac{L''}{L} Q_2 + 4\frac{L'''}{L} Q_1 + \frac{L^{(4)}}{L} Q_0 \\ &= P^2 - P'' - 4P'\frac{L'}{L} - 6P\frac{L''}{L} + \frac{L^{(4)}}{L} \end{aligned}$$

and so

$$A = P'' - P^2 + 4P'\frac{L'}{L} + 6P\frac{L''}{L} - \frac{L^{(4)}}{L}. \tag{31}$$

Differentiating (27) and dividing by L gives

$$\frac{L^{(4)}}{L} = P\frac{L''}{L} + \frac{3}{2}P'\frac{L'}{L} + \frac{1}{2}P''.$$

By substituting this in (31), we get

$$A = P'' - P^2 + 4P'\frac{L'}{L} + 6P\frac{L''}{L} - P\frac{L''}{L} - \frac{3}{2}P'\frac{L'}{L} - \frac{1}{2}P''$$

and so

$$A = 5P\frac{L''}{L} + \frac{5}{2}P'\frac{L'}{L} + \frac{1}{2}P'' - P^2.$$

Then we can write

$$\begin{aligned} A &= 5P(S' + S^2) + \frac{5}{2}P'S + \frac{1}{2}P'' - P^2 \\ &= 5P\left(\frac{P - S^2}{2} + S^2\right) + \frac{5}{2}P'S + \frac{1}{2}P'' - P^2 \\ &= \frac{5}{2}P^2 - \frac{5}{2}PS^2 + 5PS^2 + \frac{5}{2}P'S + \frac{1}{2}P'' - P^2 \\ &= \frac{3}{2}P^2 + \frac{5}{2}PS^2 + \frac{1}{2}P'' + \frac{5}{2}P'S. \end{aligned}$$

Finally, A is a rational function and so a polynomial. □

Example 7.1 *Suppose that $w'' = -Pw$ and P is a constant. Then*

$$\begin{aligned} w^{(4)} &= -Pw'' = P^2w, \\ w^{(6)} &= P^2w'' = -P^3w, \\ w^{(8)} &= -P^3w'' = P^4w. \end{aligned}$$

So, $v^{(k)} + Av = 0$, $v = w$, is possible for all even k, where

$$A = (-1)^{1+k/2}P^{k/2}$$

is also a constant.

References

[1] A. Alotaibi. On complex oscillation theory. *Results in Mathematics*, 47:165–175, 2005.

[2] S. B. Bank and I. Laine. On the oscillation theory of $f'' + Af = 0$ where A is entire. *Transactions of the American Mathematical Society*, 273(1):351–363, 1982.

[3] S. B. Bank and I Laine. Representations of solutions of periodic second order linear differential equations. *J. reine angew. Math.*, 344:1–21, 1983.

[4] S. B. Bank, I Laine, and J. K. Langley. On the frequency of zeros of solutions of second order linear differential equations. *Results in Mathematics*, 10:8–24, 1986.

[5] S. B. Bank, I Laine, and J. K. Langley. Oscillation results for solutions of linear differential equations in the complex plane. *Results in Mathematics*, 16:3–15, 1989.

[6] S. B. Bank and J. K. Langley. Oscillation theory for higher order linear differential equations with entire coefficients. *Complex Variables Theory Appl.*, 16(2-3):163–175, 1991.

[7] W. K. Hayman. *Meromorphic functions.* Oxford Mathematical Monographs. Clarendon Press, Oxford, 1964.

[8] W. K. Hayman. The local growth of power series: a survey of the Wiman-Valiron method. *Canad. Math. Bull*, 17(3):317–358, 1974.

[9] E. Hille. *Ordinary differential equations in the complex domain.* Dover Publications Inc., Mineola, NY, 1997. Reprint of the 1976 original.

[10] I. Laine. *Nevanlinna theory and complex differential equations.* de Gruyter Studies in Mathematics 15. Walter de Gruyter & Co., Berlin, 1993.

[11] J. K. Langley. Some oscillation theorems for higher order linear differential equations with entire coefficients of small growth. *Results in Mathematics*, 20(1-2):517–529, 1991.

[12] J. K. Langley. On entire solutions of linear differential equations with one dominant coefficient. *Analysis*, 15(2):187–204, 1995. Corrections: Analysis 15 (1995), 433.

[13] J. Rossi. Second order differential equations with transcendental coefficients. *Proc. American Mathematical Society*, 97:61–66, 1986.

[14] H. Wittich. Neuere Untersuchungen über eindeutige analytische Funktionen. *Ergebnisse der Mathematik und ihrer Grenzgebiete 8 (Springer, Berlin, Göttingen, Heidelberg)*, 1955.

Asim Asiri

Current address:
School of Mathematical Sciences, University of Nottingham, NG7 2RD, UK.
Current email: pmxamka@nottingham.ac.uk

Permanent affiliation: King Abdulaziz University, Jeddah, Saudi Arabia.
Permanent email: amkasiri@kau.edu.sa